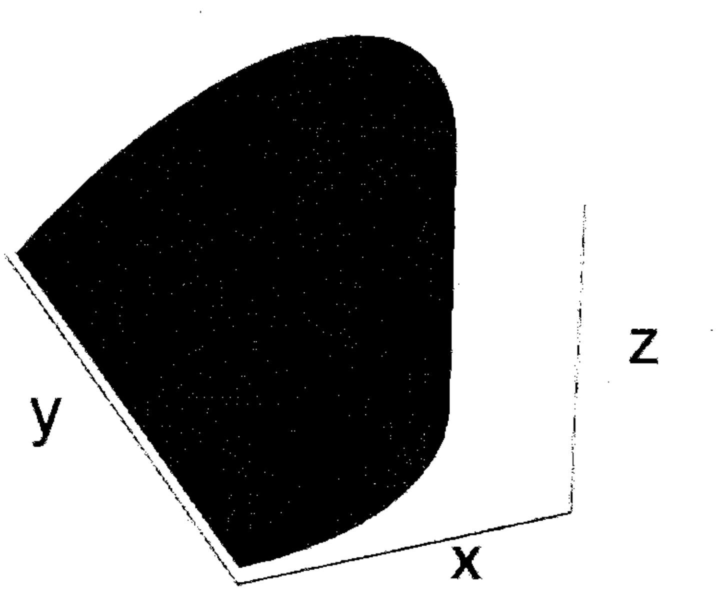
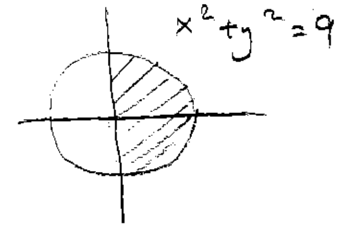
Entry Task (Old Exam Question)

Find the volume of the wedge shaped solid that lies above the xy-plane, below the plane $z = x$, and within the solid cylinder $x^2 + y^2 \leq 9$.



$$\iint_D x \, dA$$

$$\int_{-\pi/2}^{\pi/2} \int_0^3 r\cos\theta \, r \, dr \, d\theta$$



$$\int_{-\pi/2}^{\pi/2} \cos\theta \ \frac{1}{3} r^3 \Big|_0^3 \, d\theta$$

$$= 9 \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta$$

$$= 9 \left(\sin\theta \Big|_{-\pi/2}^{\pi/2} \right)$$

$$= 9(1 - -1) = \boxed{18}$$

15.4 Center of Mass

New App: Consider a thin plate (*lamina*) with density at each point given by

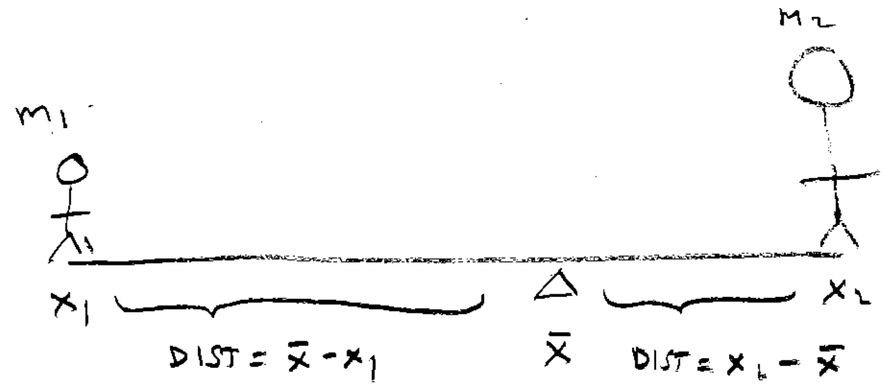
$\rho(x,y)$ = mass/area (kg/m²).

We will see that the center of mass (centroid) is given by

$$\bar{x} = \frac{\text{"Moment about y"}}{\text{Total Mass}} = \frac{\iint_R x\,p(x,y)\,dA}{\iint_R p(x,y)\,dA}$$

$$\bar{y} = \frac{\text{"Moment about x"}}{\text{Total Mass}} = \frac{\iint_R y\,p(x,y)\,dA}{\iint_R p(x,y)\,dA}$$

Motivation "the see-saw"



LAW OF LEVEL $\Rightarrow m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$

$\Rightarrow m_1\bar{x} - m_1x_1 = m_2x_2 - m_2\bar{x}$

$\Rightarrow m_1\bar{x} + m_2\bar{x} = m_1x_1 + m_2x_2$

$\Rightarrow (m_1 + m_2)\bar{x} = m_1x_1 + m_2x_2$

$$\bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

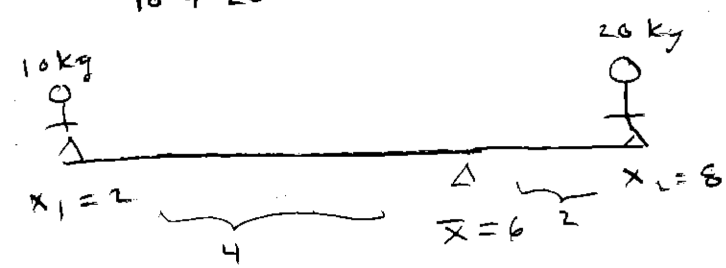
"MOMENT ABOUT y-AXIS"

<u>Ex)</u> $m_1 = 10$, $m_2 = 20$

$x_1 = 2$, $x_2 = 8$

$$\bar{x} = \frac{(10)(2) + (20)(8)}{10 + 20} = \frac{20 + 160}{30} = \frac{180}{30} = 6$$

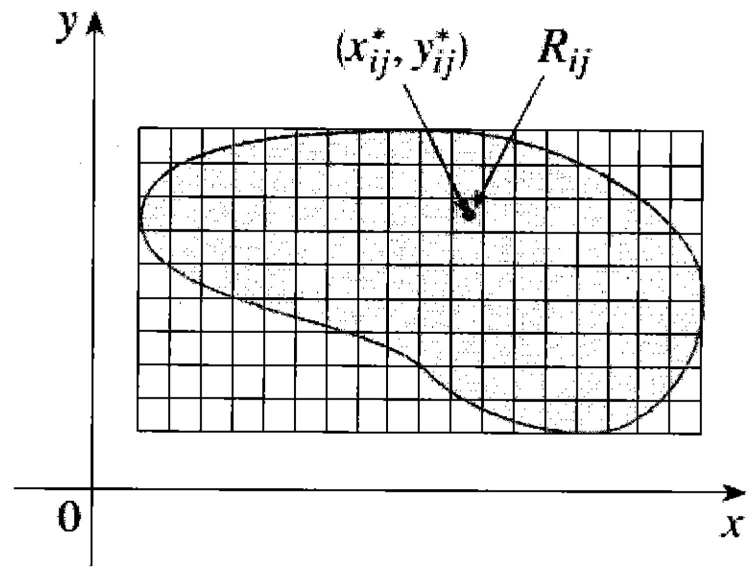
"TOTAL MASS"



In general: If you are given ***n* points** $(x_1,y_1), (x_2,y_2), \ldots, (x_n,y_n)$ with corresponding masses $m_1, m_2, \ldots, m_n$ then

$$\bar{x} = \frac{m_1x_1 + \cdots + m_nx_n}{m_1 + \cdots + m_n} = \frac{M_y}{M}$$

$$\bar{y} = \frac{m_1y_1 + \cdots + m_ny_n}{m_1 + \cdots + m_n} = \frac{M_x}{M}$$



Derivation:

1. Break region into m rows and n columns.
2. Find center of mass of each rectangle:
$$(\bar{x}_{ij}, \bar{y}_{ij})$$
3. Estimate the mass of each rectangle:
$$m_{ij} = p(\bar{x}_{ij}, \bar{y}_{ij})\Delta A$$
4. Now use the formula for *n* points.
5. Take the limit.

$$\bar{x} = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}}$$
$$= \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} \bar{x}_{ij}\, p(\bar{x}_{ij}, \bar{y}_{ij})\Delta A}{\sum_{i=1}^{m}\sum_{j=1}^{n} p(\bar{x}_{ij}, \bar{y}_{ij})\Delta A}$$

Center of Mass:

$$\bar{x} = \frac{\text{Moment about y}}{\text{Total Mass}} = \frac{\iint_R \, x\, p(x,y)dA}{\iint_R \, p(x,y)dA}$$

$$\bar{y} = \frac{\text{Moment about x}}{\text{Total Mass}} = \frac{\iint_R \, y\, p(x,y)dA}{\iint_R \, p(x,y)dA}$$

Example:
Consider a 1 by 1 m square metal plate. The density is given by *p(x,y) = kx* kg/m^2 for some constant *k*.
Find the center of mass.

Side note:
The density *p(x,y) = kx* means that the density is proportional to *x* which can be thought of as distance from the *y*-axis. In other words, the plate gets heavier at a constant rate from left-to-right.

TOTAL MASS $= \int_0^1 \int_0^1 kx\,dx\,dy = \int_0^1 \frac{k}{2}x^2\Big|_0^1 dy$
$= \frac{k}{2}\int_0^1 1\,dy = \frac{k}{2}\,y\Big|_0^1 = \boxed{k/2 = M}$

MOMENT ABOUT y $= \int_0^1 \int_0^1 x\,(kx)\,dx\,dy = \int_0^1 \frac{k}{3}x^3\Big|_0^1 dy$
$= \frac{k}{3}\int_0^1 1\,dy = \boxed{\frac{k}{3} = M_y}$

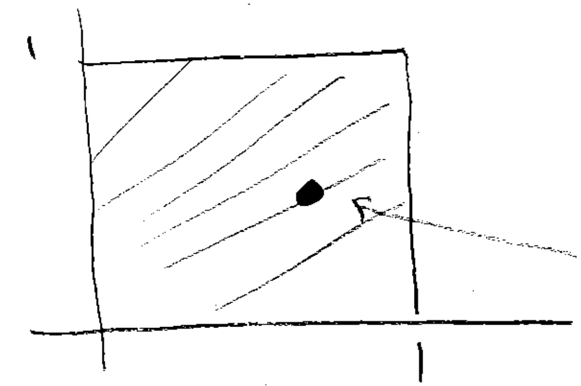
MOMENT ABOUT x $= \int_0^1 \int_0^1 y\,(kx)\,dx\,dy = \int_0^1 \frac{k}{2}\,y\,x^2\Big|_0^1 dy$
$= \frac{k}{2}\int_0^1 y\,dy = \frac{k}{2}\,\frac{1}{2}y^2\Big|_0^1 = \boxed{\frac{k}{4} = m_x}$

$\bar{x} = \frac{M_y}{m} = \frac{k/3}{k/2} = 2/3$

$\bar{y} = \frac{m_x}{m} = \frac{k/4}{k/2} = 1/2$

CENTER OF MASS

$\boxed{\left(\frac{2}{3}, \frac{1}{2}\right)}$



Example If $k = 15$
then $p(\frac{1}{3},1) = 15 \cdot \frac{1}{3} = 5 \ \frac{kg}{m^2}$
$p(\frac{2}{3},1) = 15 \cdot \frac{2}{3} = 10 \ \frac{kg}{m^2}$
$p(1,1) = 15 \cdot 1 = 15 \ \frac{kg}{m^2}$

SAME ANSWER NO MATTER WHAT K IS.

Translations:

Density proportional to the dist. from...

...the y-axis -- $p(x,y) = kx.$

...the x-axis -- $p(x,y) = ky.$

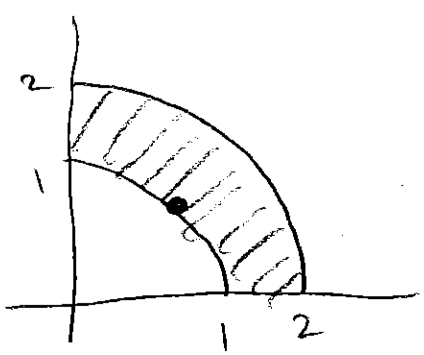
...the origin -- $p(x,y) = k\sqrt{x^2+y^2}.$

Density proportional to the <u>square</u> of the distance from the origin:

$$p(x,y) = k(x^2+y^2).$$

Density <u>inversely</u> proportional to the distance from the origin:

$$p(x,y) = \frac{k}{\sqrt{x^2+y^2}}$$



Example: A thin plate is in the shape of the region bounded between the circles of radius 1 and 2 in the first quadrant. The density is proportional to the distance from the origin.
Find the center of mass.

$$p(x,y) = k\sqrt{x^2+y^2}$$

$$M = \iint_D p(x,y)\,dA = \int_0^{\pi/2}\int_1^2 kr \cdot r\,dr\,d\theta$$
$$= \int_0^{\pi/2} \frac{k}{3} r^3 \Big|_1^2 \, d\theta$$
$$= \int_0^{\pi/2} \frac{k}{3}(8-1)\,d\theta$$
$$= \frac{7}{3}k\ \theta\Big|_0^{\pi/2} = \frac{7\pi}{6}k$$

$$M_y = \iint_D x\,p(x,y)\,dA = \int_0^{\pi/2}\int_1^2 r\cos\theta\ kr \cdot r\,dr\,d\theta$$
$$= k\int_0^{\pi/2} \cos\theta\ \frac{1}{4}r^4\Big|_1^2 \, d\theta = \frac{k}{4}(16-1)\ \sin\theta\Big|_0^{\pi/2}$$
$$= \frac{15k}{4}$$

$$M_x = \iint_D y\,p(x,y)\,dA = \int_0^{\pi/2}\int_1^2 r\sin\theta\ kr\ r\,dr\,d\theta$$
$$= k\int_0^{\pi/2} \sin\theta\ \frac{1}{4}r^4\Big|_1^2 \, d\theta = \frac{k}{4}(16-1)\left(-\cos\theta\Big|_0^{\pi/2}\right) = \frac{15k}{4}$$

$$\bar{x} = \frac{15k/4}{7\pi k/6} = \frac{45}{14\pi} \approx 1.02313 \qquad \bar{y} = \frac{15k/4}{7\pi k/6} = \frac{45}{14\pi} \approx 1.02313$$

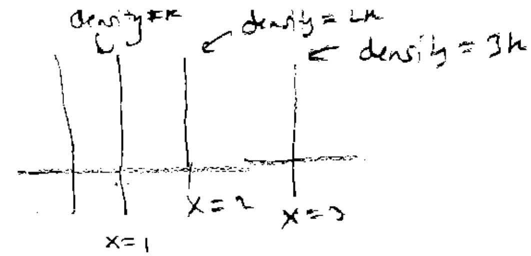
Translations:

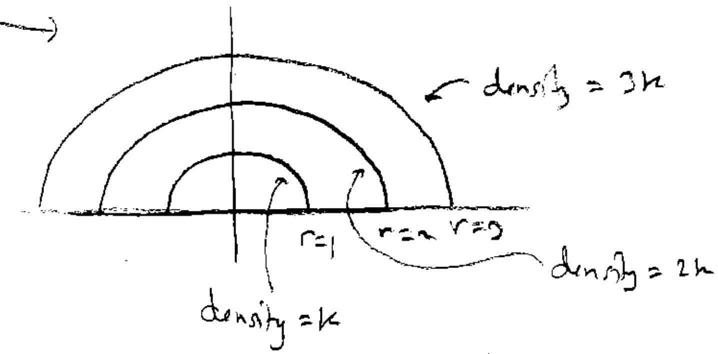
Density proportional to the dist. from…

…the y-axis -- $p(x,y) = kx.$

…the x-axis -- $p(x,y) = ky.$

…the origin -- $p(x,y) = k\sqrt{x^2+y^2}.$



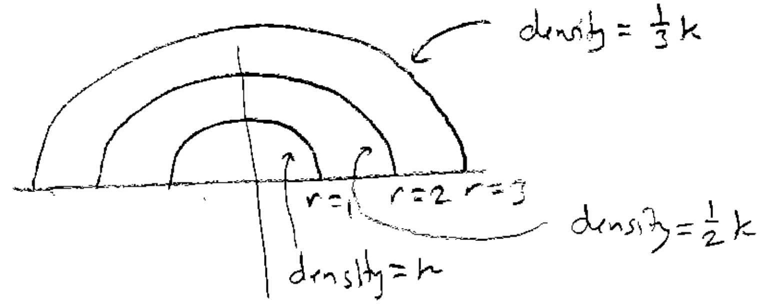


Density proportional to the *<u>square</u>* *of the distance from the origin*:

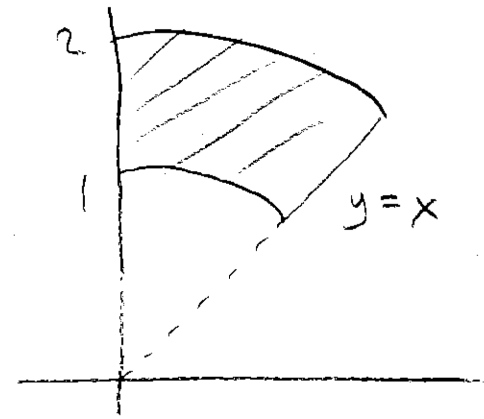
$$p(x,y) = k(x^2+y^2).$$

Density *<u>inversely</u>* *proportional to the distance from the origin*:

$$p(x,y) = \frac{k}{\sqrt{x^2+y^2}}$$



Example (Old Exam Question)
A lamina occupies the region R in the first quadrant that is above the line $y = x$ and between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.
The density is proportional to the distance from the origin.
Find the y-coordinate of the center of mass.



$$\rho(x,y) = k\sqrt{x^2+y^2}$$

$$\bar{y} = \frac{\iint_D y\,\rho(x,y)\,dA \leftarrow \boxed{\text{I}}}{\iint_D \rho(x,y)\,dA \leftarrow \boxed{\text{II}}}$$

$$\boxed{\text{I}} = \int_{\pi/4}^{\pi/2}\int_1^2 r\sin\theta\; k\, r\cdot r\,dr\,d\theta$$
$$= k\int_{\pi/4}^{\pi/2}\sin\theta\int_1^2 r^3\,dr\,d\theta$$
$$= k\int_{\pi/4}^{\pi/2}\sin\theta\;\tfrac{1}{4}r^4\Big|_1^2\,d\theta$$
$$= \frac{k}{4}(16-1)\int_{\pi/4}^{\pi/2}\sin\theta\,d\theta$$
$$= \frac{15k}{4}\left(-\cos(\theta)\Big|_{\pi/4}^{\pi/2}\right)$$
$$= \frac{15k}{4}\left(0 - -\tfrac{\sqrt{2}}{2}\right) = \frac{15\sqrt{2}\,k}{8}$$

$$\boxed{\text{II}}\quad \int_{\pi/4}^{\pi/2}\int_1^2 k\,r\cdot r\,dr\,d\theta$$
$$= k\int_{\pi/4}^{\pi/2}\tfrac{1}{3}r^3\Big|_1^2\,d\theta$$
$$= \frac{k}{3}(8-1)\int_{\pi/4}^{\pi/2}1\,d\theta$$
$$= \frac{7k}{3}\;\theta\Big|_{\pi/4}^{\pi/2}$$
$$= \frac{7k}{3}\left(\tfrac{\pi}{2}-\tfrac{\pi}{4}\right)$$
$$= \frac{7\pi k}{12}$$